Interference and Throughput in Aloha-based Ad Hoc Networks with Isotropic Node Distribution

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Abstract—**We study the interference and outage statistics in a slotted Aloha ad hoc network, where the spatial distribution of nodes is non-stationary and isotropic. In such a network, outage probability and local throughput depend on both the particular location in the network and the shape of the spatial distribution. We derive in closed-form certain distributional properties of the interference that are important for analyzing wireless networks as a function of the location and the spatial shape. Our results focus on path loss exponents 2 and 4, the former case not being analyzable before due to the stationarity assumption of the spatial node distribution. We propose two metrics for measuring local throughput in non-stationary networks and discuss how our findings can be applied to both analysis and optimization.**

Index Terms—**Ad hoc networks, interference, throughput**

I. INTRODUCTION

The application of stochastic geometry to the modeling and analysis of wireless networks has attained a lot of attention in recent years. It has enabled a new framework called *transmission capacity* (TC) framework, which led to many new profound results in the topic of wireless networks (cf. [1], [2]). The advantage of using a spatial model to describe the node positions rather than assuming a deterministic network topology is two-fold: First, such a probabilistic approach decouples the performance analysis from the *actual* topology, thereby increasing the generality of results. Second, it provides powerful means for network optimization, especially for highly dynamic networks, where interference is (unpredictably) fast-varying.

With few exceptions, the node positions are mostly modeled as a *stationary* point process. Stationarity is a desirable property, allowing analytically tractable computations and, even more important, representing a key requirement for applying the definition of TC. Even though the stationarity assumption has not really narrowed the range of obtainable insights, it poses some shortcomings to the analysis of wireless networks:

Infinite networks: Stationarity implies that the network is infinitely large as opposed to real deployments with a finite number of nodes.

No border effects: Border effects are inherently neglected in infinite networks. However, border effects cause heterogeneity in the nodes' capabilities depending on their location, i.e., being dis-/connected, interference-/noise-limited, etc.

Infinite interference for free-space path loss: For stationary node distributions in the plane and a path loss exponent $\alpha = 2$, the interference is infinite almost sure (a.s.) [1], resulting in a TC of zero. More specifically, stationary models lose their accuracy as the path loss exponent decreases due to the fact that infinitely many nodes contribute to the interference.

Application of the TC: As already mentioned, the TC applies only to stationary networks. When the node distribution is non-stationary, this metric must be modified to take into account heterogeneous node deployments.

In reality, wireless ad hoc networks always exhibit a heterogeneous node distribution. The most obvious example is perhaps when the nodes are distributed in a *bounded* region. In such a network, the interference situation near the center will significantly differ from that at the border. Besides this simple example, more complex deployments are often found in practice, e.g., wireless sensor networks created by airdrop [3], spontaneous formation of hot spots [4], etc.

A. Contribution

We extend the existing framework by relaxing the requirements on the node distribution, thereby allowing for *isotropy only*. More specifically, we have the following results:

- The interference and outage statistics for slotted Aloha with $\alpha = 2$ and $\alpha = 4$ are derived as a function of the receiver position and the spatial shape of the node distribution. We consider a path loss plus block fading model. As for the outage statistics, we focus on Rayleigh fading. We show how known results for the stationary case arise from our results as special cases.
- Two global metrics, namely the differential TC and the average sum throughput, that take into account heterogeneous node deployments are proposed. While the former metric is a refinement of the TC, the latter quantifies the first order overall network efficiency.

B. Related work

Stationary models with heterogeneous node deployment have already been investigated. Specifically, Poisson-Cluster [5] and Matérn hard-core models [6] have been studied, as they are well-suited for analyzing more sophisticated medium access control (MAC) schemes. Treated as *general motion-invariant*, these and similar models were further analyzed in [2], [7], [8] in a unifying way. In [9], a non-stationary and isotropic node distribution was assumed for analyzing multi-antenna receivers. While the analysis showed that the shape of the spatial distribution has a considerable impact on link performance, the scenario was limited only to the case of the receiver located in the origin.

II. Network model

We consider a wireless ad hoc network with nodes *isotropically* distributed in $\mathbb{R}^2$. The MAC employed by the nodes is slotted Aloha. In a randomly chosen slot, some nodes wish to transmit a packet. We assume that the set of transmitters $\{x\}$ follows an isotropic Poisson point process (PPP) $\Phi_t := \{x\}$ on $\mathbb{R}^2$ with intensity $\lambda(x)$, where $x \in \mathbb{R}^2$. Due to the isotropy of Φ_t, $\lambda(x)$ is rotation-invariant and depends only on the Euclidean norm $\|x\|$, i.e., $\lambda(x) = \lambda(\|x\|e^{j\phi}) = \lambda(\|x\|)$, $\phi \in [0, 2\pi)$. When working with polar coordinates, we will use the notation $\lambda(r)$, where $r := \|x\|$. With [6], we can describe $\lambda(r)$ as the resulting intensity after *distance-dependent* thinning of a stationary PPP of intensity λ, i.e.,

$$\lambda(r) := \lambda F(r), \tag{1}$$

where $F(r)$ is called the *shape function* as it reflects the spatial shape of Φ_t. We will pose the following restrictions on $F(r)$:

(i) Positiveness: $F(r) \geq 0$ for all $r \geq 0$.
(ii) Normalization: $\max_r \{F(r)\} = 1$.

The restrictions (i) and (ii) are necessary to ensure that $\lambda(r)$ is non-negative and bounded by λ everywhere.

We assume that each transmitter x has an intended receiver y randomly located at fixed distance d. From the random translation Theorem [6] it follows that the set of receivers $\{y\}$ forms an isotropic PPP $\Phi_r := \{y\}$ on $\mathbb{R}^2$ with intensity $\lambda(x)$ as well. The fixed distance assumption is commonly accepted, see [1]. However, we will relax this assumption in Section V.

We consider a path loss plus block fading channel with independent and identically distributed (i.i.d.) fading coefficients. The power path loss between two positions $x, y \in \mathbb{R}^2$ is given by $\ell(\|x - y\|) := (c + \|x - y\|^\alpha)^{-1}$ with path loss exponent α. The parameter $c > 0$ ensures boundedness of ℓ. The power fading coefficient between a transmitter at x and a receiver at y is given by g_{xy}, where $\mathbb{E}[g_{xy}] = 1$ for all $x, y \in \mathbb{R}^2$.

We further place a receiver at $y_0 \in \mathbb{R}^2$ and an intended transmitter at an arbitrary position $x_0 \in \mathbb{R}^2$ with distance d to y_0. The pair $x_0 \rightarrow y_0$ is called the *reference pair* as it will allow us to measure the (spatially-averaged) link performance for receivers at distance $\|y_0\|$ from the origin.

Assuming fixed power transmissions for all nodes, the instantaneous signal-to-interference-plus-noise ratio (SINR) at the reference receiver y_0 is given by:

$$\mathsf{SINR}(y_0) := \frac{g_{x_0 y_0}}{\eta + \ell(d)^{-1}\mathsf{I}(y_0)}, \tag{2}$$

where η is the average noise-to-signal ratio and

$$\mathsf{I}(y_0) := \sum_{x \in \Phi_t \setminus \{x_0\}} g_{xy_0} \ell(\|x - y_0\|) \tag{3}$$

is the interference power. We assume strong channel coding, i.e., the outage event is a steep function of the SINR. The outage probability (OP) at the reference pair $x_0 \rightarrow y_0$ is then given by the reduced Palm probability

$$q(y_0) := \mathbb{P}^{!x_0}\left(\mathsf{SINR}(y_0) < \beta\right), \tag{4}$$

where β is a modulation and coding specific SINR threshold.

III. Interference analysis

We now study the interference statistics at the reference receiver at y_0. Before, we note the following two integral identities which are taken from [10]:

Identity 1. If $a > |b|$, $a, b \in \mathbb{R}$,

$$\int_0^\pi \frac{\mathrm{d}\phi}{(a + b\cos\phi)^{n+1}} = \frac{\pi P_n\left(\frac{a}{\sqrt{a^2 - b^2}}\right)}{(a^2 - b^2)^{\frac{n+1}{2}}}, \tag{5}$$

where $P_n(x)$ is the nth-Legendre polynomial.

Identity 2. Let $a_1, a_2, a_3 \in \mathbb{R}$, $R := a_1 + a_2 t^2 + a_3 t^4$, $\Delta = 4a_1 a_3 - a_2^2$ and $a_3 > 0$. Using substitution $t \rightarrow t^2$, we have

$$\int \frac{2t\sqrt{a_3}\,\mathrm{d}t}{\sqrt{a_1 + a_2 t^2 + a_3 t^4}} = \begin{cases} \log\frac{2\sqrt{a_3 R} + 2a_3 t^2 + a_2}{\sqrt{\Delta}}, & a_3 > 0 \\ \operatorname{arcsinh}\frac{2a_3 t^2 + a_2}{\sqrt{\Delta}}, & \Delta > 0 \\ \log(2a_3 t^2 + a_2), & \Delta = 0. \end{cases} \tag{6}$$

We are now in the position to derive the first moment of the interference at y_0.

Theorem 1. Let $f(r) := \mathrm{d}F(r)/\mathrm{d}r$, $c > 0$ and $\alpha = 2$. If $\lim_{r \to \infty} F(r)r^\nu < \infty$ for some $\nu > 0$, then

$$\mathbb{E}^{!x_0}[\mathsf{I}(y_0)] = \lambda A_2(y_0, c) < \infty, \tag{7}$$

where the *interference-driving* function $A_2(y_0, c)$ is given by

$$A_2(y_0, c) := F(0)\operatorname{arcsinh}\frac{y_0^2 - c}{2y_0\sqrt{c}}$$
$$+ \int_0^\infty f(r)\operatorname{arcsinh}\frac{y_0^2 - r^2 - c}{2y_0\sqrt{c}}\,\mathrm{d}r. \tag{8}$$

Proof: We write

$$\mathbb{E}^{!x_0}[\mathsf{I}(y_0)] = \lambda \int_{\mathbb{R}^2} \mathbb{E}[g_{xy_0}]\ell(\|x - y_0\|)F(x)\,\mathrm{d}x$$

what follows from Campbell's Theorem and Slivnyak's Theorem [11], and from the i.i.d. property of g_{xy_0}. Applying Identity 1 and 2 yields the result. ∎

The function $A_2(y_0, c)$ in (8) has an interesting interpretation: $A_2(y_0, c)$ can be described as the interference field associated with the origin o, from which the remaining interference adds up differentially.

Corollary 1. Summary of some special cases of Theorem 1:

1) When we assume $F(0) = 1$ and $f(r) \leq 0$ for all $r \in \mathbb{R}_+$, $F(r)$ can be interpreted as a complementary cumulative distribution function (CDF) with respect to a *random* distance r to the origin, yielding

$$A_2(y_0, c) = \operatorname{arcsinh}\frac{y_0^2 - c}{2y_0\sqrt{c}} - \mathbb{E}\left[\operatorname{arcsinh}\frac{y_0^2 - \mathsf{r}^2 - c}{2y_0\sqrt{c}}\right].$$

2) Letting $\|y_0\| \rightarrow 0$, we further have

$$A_2(0, c) = \log(1/2c) + \mathbb{E}[\log(2(\mathsf{r} + c))].$$

3) Letting $c \rightarrow 0$, we have $\mathbb{E}[\mathsf{I}(y_0)] = \infty$, which is due to the resulting singularity of $\ell(\|x - y_0\|)$ at $x = y_0$, cf. [2].

4) Sparse network ($0 < \lim_{r\to\infty} F(r)r^\nu < \infty$, $0 < \nu \leq 2$): Remarkably, $\int_0^\infty r F(r)\mathrm{d}r = \infty$ but $\mathbb{E}^{!x_0}[\mathsf{I}(y_0)] < \infty$.
5) Dense network ($0 < \lim_{r\to\infty} F(r)r^\nu < \infty$, $\nu \to 0$): As expected [2], $\mathbb{E}^{!x_0}[\mathsf{I}(y_0)] = \infty$.

1) has an interesting interpretation as well: The expectation can be seen as *averaging* the differential interference over r. Such an interpretation may be appropriate when analyzing networks with a priori unknown or fast-varying spatial configurations, for which a CDF is then used to model their spatial shape. 4) implies $\mathsf{I}(y_0) < \infty$ a.s. although infinitely many nodes contribute to the interference on average. Note that 5) includes the homogeneous case with $F(r) = 1$ ($f(r) = 0$).

We now extend the findings of Theorem 1. Before, we need the following Lemma.

Lemma 1. Let $a_1, a_2 \in \mathbb{R}$, $a_1 > 0$. Then,

$$\int\int_0^\pi \frac{2t\,\mathrm{d}\phi\,\mathrm{d}t}{a_1 + (t^2 + a_2^2 - 2ta_2\cos\phi)^2} = \frac{\pi}{2\sqrt{a_1}}\arctan\frac{2\mathrm{Re}\{\kappa(t,a_1,a_2)\}}{1 - |\kappa(t,a_1,a_2)|^2}, \quad (9)$$

where

$$\kappa(t,a_1,a_2) := \frac{t^2 - a_2^2 - j\sqrt{a_1}}{\sqrt{(\sqrt{a_1} + j(t^2 + a_2^2))^2 + 4t^2 a_2^2}}. \quad (10)$$

Proof: The basic idea is to decompose the integrand into partial fractions and to apply Identity 1 and 2, yielding (9) after some algebraic manipulations. Note that according to [10], (5) and (6) hold only for real-valued parameters. However, they were verified to hold also for complex-valued parameters. ∎

Theorem 2. Let $f(r) := \mathrm{d}F(r)/\mathrm{d}r$, $c > 0$ and $\alpha = 4$. Then,

$$\mathbb{E}^{!x_0}[\mathsf{I}(y_0)] = \lambda A_4(y_0,c) < \infty, \quad (11)$$

where $A_4(y_0,c)$ is given by (12) below.

Proof: The proof is analogous to the proof of Theorem 1 and uses the integral identity of Lemma 1. We further make use of (ii) in Section II to show that $\lim_{r\to\infty} F(r) < \infty$. ∎

Corollary 2. Summary of some special cases of Theorem 2:
1) Case $c \to 0$: By taking the limit $\lim_{c\to 0} A_4(y_0,c)$ in (12), we observe that $\mathbb{E}^{!x_0}[\mathsf{I}(y_0)] = \infty$, cf. 3) in Corollary 2.
2) Homogeneous case: Let $F(r) = 1$. Then, $f(r) = 0$ and

$$\lim_{r\to a}\arctan\frac{2\mathrm{Re}\{\kappa(r,c,y_0)\}}{1 - |\kappa(r,c,y_0)|^2} = \begin{cases} -\frac{\pi}{2}, & a = 0 \\ \frac{\pi}{2}, & a = \infty, \end{cases}$$

yielding $\mathbb{E}^{!x_0}[\mathsf{I}(y_0)] = \lambda\frac{\pi^2}{2\sqrt{c}}$ as expected, cf. [2].
All results of this Corollary are consistent with the literature.

The first moment of the interference is useful for bounding the interference distribution for the path loss only scenario.

In case of Rayleigh fading channels, the Laplace transform of $\mathsf{I}(y_0)$, i.e., $\mathcal{L}_{\mathsf{I}(y_0)}(s) := \mathbb{E}[\exp\{-s\mathsf{I}(y_0)\}]$, is of significant importance, since it allows one to obtain the OP in closed-form. When treating the case $\alpha = 2$, we will always assume that $F(r)$ satisfies the additional condition of Theorem 1.

Theorem 3. For $\mathsf{g}_{xy} \sim \mathrm{Exp}(1)$ for all $x, y \in \mathbb{R}^2$ (Rayleigh fading), the Laplace transform of $\mathsf{I}(y_0)$ at $y_0 \in \mathbb{R}^2$ is

$$\mathcal{L}_{\mathsf{I}(y_0)}(s) = \exp\{-\lambda s\, A_\alpha(y_0, s+c)\}, \quad (13)$$

for the cases $\alpha = 2$ and $\alpha = 4$, where $A_2(y_0, c)$ is given by (8) and $A_4(y_0, c)$ is given by (12).

Proof: We write

$$\mathcal{L}_{\mathsf{I}(y_0)}(s) \stackrel{(a)}{=} \mathbb{E}^{!x_0}_{\Phi_t}\left[\prod_{\mathsf{x}\in\Phi_t}\mathbb{E}_{\mathsf{g}_{\mathsf{x}y_0}}[\exp\{-s\mathsf{g}_{\mathsf{x}y_0}\ell(\|\mathsf{x}-y_0\|)\}]\right]$$

$$\stackrel{(b)}{=} \exp\left\{-\int_{\mathbb{R}^2}(1 - \mathcal{L}_{\mathsf{g}}(s\ell(\|x-y_0\|)))\,\lambda(x)\mathrm{d}x\right\},$$

where (a) follows from algebraic manipulations and the i.i.d. property of the $\mathsf{g}_{\mathsf{x}y_0}$. (b) follows from the probability generating functional and the Laplace functional of a PPP [6]. After noting that $\mathcal{L}_{\mathsf{g}}(s) = (1+s)^{-1}$ for $\mathsf{g} \sim \mathrm{Exp}(1)$, the integral is computed using Identity 1 and 2, and Lemma 1. ∎

Note that (a) in the proof holds for general point processes and some approximation techniques for computing the right-hand side already exist [12]. The (b) part is for PPPs only.

Corollary 3. Setting $F(r) = 1$ for all $r \in \mathbb{R}_+$ and $c = 0$, we obtain the well-known result for the homogeneous case with $\alpha = 4$ [2]: $\mathcal{L}_{\mathsf{I}(y_0)}(s) = \exp\{-\lambda\frac{\pi^2}{2}\sqrt{s}\}$.

IV. OUTAGE AND LOCAL THROUGHPUT

A. Outage probability

We now study the OP for the reference pair $x_0 \to y_0$. In order to broadly discuss the impact of the spatial shape on the performance, we focus on the Rayleigh fading scenario. For other channel models, the interference moments derived in Section III can be used to effectively bound the OP, e.g., using the Markov inequality [1]. We do not expect additional insights by considering also other channel models.

Theorem 4. The OP for the Rayleigh fading scenario and $\alpha = 2$ respectively $\alpha = 4$ is given by

$$q(y_0) = 1 - \mathcal{L}_{\mathsf{I}(y_0)}(\beta(c+d^\alpha))\exp\{-\beta\eta\}. \quad (14)$$

Proof: It is well-known that the OP for Aloha MAC and exponentially distributed power gains g_{xy} can be written in terms of the Laplace transform of the interference [2], [6]: We condition (2) on Φ and evaluate the OP first with respect to $\mathsf{g}_{x_0 y_0}$. We finally use (13) with $s = \beta(c+d^\alpha)$. ∎

By means of (14) in Theorem 4 we can now measure the OP for Rayleigh fading at an arbitrary location for an arbitrary

$$A_4(y_0,c) := \frac{\pi}{2\sqrt{c}}\left(F(r)\arctan\frac{2\mathrm{Re}\{\kappa(r,c,y_0)\}}{1 - |\kappa(r,c,y_0)|^2}\Big|_{r=0}^\infty - \int_0^\infty f(r)\arctan\frac{2\mathrm{Re}\{\kappa(r,c,y_0)\}}{1 - |\kappa(r,c,y_0)|^2}\,\mathrm{d}r\right) \quad (12)$$

spatial shape function $F(r)$ satisfying the given restrictions. Fig. 1 shows $q(\|y_0\|)$ vs. $\|y_0\|$ for $\alpha = 2$ and $\alpha = 4$, thereby confirming the analysis. It can further be observed how the network "moves" from the interference-limited to the power-limited regime with increasing $\|y_0\|$.

To highlight the accuracy of the model, we compare the OP from Theorem 4 to a straightforward way of approximating the OP which consists of assuming that the intensity $\lambda(x)$ is approximately constant around y_0. In this case the OP can then be described as in the homogeneous case [2], except for the intensity in the exponential term being modulated by $F(y_0)$, i.e., $\tilde{q}(y_0) := 1 - \exp\{-F(y_0)\lambda\pi^2 d^2 \beta^{\frac{2}{\alpha}} \frac{2}{\alpha} \csc \frac{2\pi}{\alpha}\} \approx q(y_0)$. We will now study the logarithmic ratio of exact to approximate success probability, i.e., $\gamma := \log \frac{1-q(y_0)}{1-\tilde{q}(y_0)}$.

Corollary 4. Let $c = 0$. The ratio γ for $\alpha = 4$ is given by

$$\gamma = \lambda d^2 \sqrt{\beta} \left(\frac{\pi^2}{2} F(y_0) - d^2 \sqrt{\beta} A_4(y_0, \beta d^4) \right). \quad (15)$$

Fig. 2 shows the ratio γ together with the shape function $F(r)$ for different receiver positions y_0. $F(r)$ was chosen such that the network exhibits a communication hotspot, with the density of active nodes slowly decaying between 70 and 500 until it becomes approximately zero. One can see that the approximation is not satisfactory, especially in the transition region, where border effects come into play.

B. Local throughput

We now propose two local throughout metrics that are suitable for non-stationary wireless ad hoc networks.

Definition 1 (Differential transmission capacity (DTC)). The DTC is defined as the maximal density of concurrent transmissions in an *infinitesimal* region around the point $x \in \mathbb{R}^2$ subject to an OP constraint ϵ, i.e.,

$$c(x, \epsilon) := \lambda(x, \epsilon)(1 - \epsilon). \quad (16)$$

The TC and its differential counterpart have similar meaning, except that the latter is *position-dependent*: For a given spatial shape $F(r)$ and target OP ϵ, $c(x, \epsilon)$ yields the TC in a region $\mathrm{d}x$. Hence, the DTC implicitly takes into account the spatial shape of the node distribution. For Rayleigh fading, $c(x, \epsilon)$ is obtained by solving (14) for λ. Like the TC, the DTC can be used for comparing different transmission protocols.

Definition 2 (Average sum throughput (AST)). The AST is defined as the ratio of average number of successful transmissions to average number of simultaneous transmissions, i.e.,

$$\Omega := \frac{\mathbb{E}\left[\sum_{x \in \Phi_t} \mathbb{1}_{\{x \text{ successful}\}} \right]}{\mathbb{E}\left[\sum_{x \in \Phi_t} \mathbb{1}_{\{x \in \mathbb{R}^2\}} \right]}. \quad (17)$$

The AST quantifies the first order overall efficiency of the network on the MAC layer. While the DTC highlights the spatial dynamics of the local throughput, the AST yields a single figure of merit. In essence, the AST counts the number of successful transmissions, thereby integrating over the spatial dynamics. Note that the success function $\mathbb{1}_{\{x \text{ successful}\}}$,
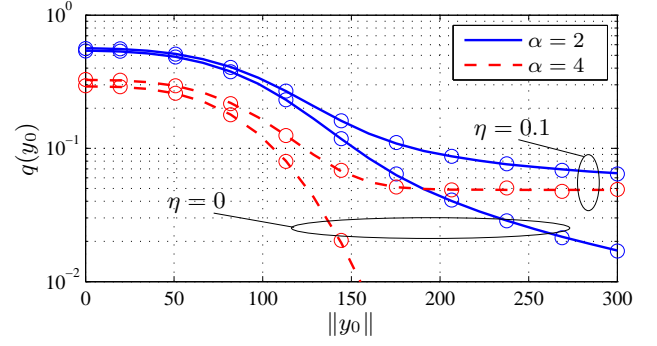

Fig. 1. $q(y_0)$ vs. $\|y_0\|$ for $F(r) = \exp\{-(r/100)^3\}$, $d = 10$, $\beta = 0.5$, $c = 1$, $\lambda = 0.001$. Marks represent the simulation results.
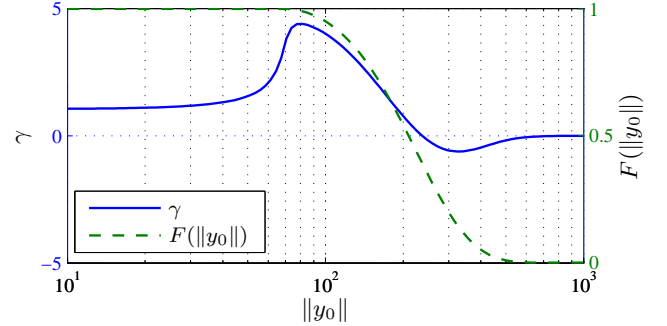

Fig. 2. γ and $F(\|y_0\|)$ vs. $\|y_0\|$ for $\alpha = 4$, $\beta = 1$, $d = 10$, $c = 1$.

indicating that transmitter x has been successful, can be chosen arbitrarily to include additional outage-inducing effects, e.g., energy-limitations, dis-connectivity or secrecy outage.

Theorem 5. Let $\lim_{r \to \infty} F(r) r^\nu < \infty$ for some $\nu > 2$. With the underlying network model and success function $\mathbb{1}_{\{\mathsf{SINR}(y) \geq \beta\}}$, the AST Ω can be computed as

$$\Omega = \frac{\int_0^\infty r(1 - q(r)) F(r) \, \mathrm{d}r}{\int_0^\infty r F(r) \, \mathrm{d}r}. \quad (18)$$

Proof: Since the denominator directly follows from Campbell's Theorem, we focus on the numerator and write

$$\mathbb{E}\left[\sum_{x \in \Phi_t} \mathbb{1}_{\{x \text{ successful}\}} \right] \overset{(a)}{=} \int_{\mathbb{R}^2} \mathbb{E}^{!x} \left[\mathbb{1}_{\{x \text{ successful}\}} \right] \lambda(x) \, \mathrm{d}x$$

$$\overset{(b)}{=} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbb{E}^{!x}_{\Phi_t} \left[\mathbb{1}_{\{\mathsf{SINR}(y) \geq \beta\}} \right] \mathbb{P}(\mathsf{y} = y|x) \, \mathrm{d}y \, \lambda(x) \, \mathrm{d}x$$

$$\overset{(c)}{=} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \mathbb{E}^{!x}_{\Phi_t} \left[\mathbb{1}_{\{\mathsf{SINR}(y) \geq \beta\}} \right] \mathbb{P}(\mathsf{y} = y|x) \, \lambda(x) \, \mathrm{d}x \right) \mathrm{d}y$$

$$\overset{(d)}{=} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \mathbb{P}^{!x}_{\Phi_t} (\mathsf{SINR}(y) \geq \beta) \mathbb{P}(\mathsf{y} = y|x) \, \lambda(x) \, \mathrm{d}x \right) \mathrm{d}y$$

$$\overset{(e)}{=} \int_{\mathbb{R}^2} (1 - q(y)) \left(\int_{\mathbb{R}^2} \mathbb{P}(\mathsf{y} = y|x) \, \lambda(x) \, \mathrm{d}x \right) \mathrm{d}y$$

$$= \int_{\mathbb{R}^2} (1 - q(y))\lambda(y) \, \mathrm{d}y.$$

(a) is due to Campbell's Theorem [11]. (b) is obtained by noting that a transmitter x is successful if the intended receiver at y is not in outage. From Section II, we know that y is placed by random translation of x according to some probability

kernel $\mathbb{P}(\mathsf{y} = y|x)$. (c) follows from Tonelli's Theorem [13] and (d) follows from $\mathbb{E}\left[\mathbb{1}_{\{X \in A\}}\right] = \mathbb{P}(X \in A)$. (e) follows from (14) and the fact that $q(y)$ is independent of x. ∎

V. APPLICATIONS OF THE MODEL

A. Shot-range inhibition

Besides slotted Aloha, other MAC protocols such as CSMA/CA or local FDMA, are promising techniques for reducing excessive interference generated by nodes within *shot-range*. To study ad hoc networks with such inhibition mechanisms while ensuring analytical tractability, powerful methods based on non-homogeneous Poisson approximation have been used [6], [14], [15]. When such protocols are *transmitter-initiated*, e.g., transmitter sensing for CSMA or transmitter orthogonalization for FDMA, the resulting spatial distribution of interferers becomes inhomogeneous and approximately isotropic around the transmitter x, while the interference field at the intended receiver y will depend on the distance $\|x - y\|$. Hence, our model can be applied also to such modeling problems and is not limited to Aloha MAC.

B. Network optimization

Consider the following situation: Let the set $\{\mathsf{y}\}$ of *potential* receivers be distributed as an isotropic PPP Φ_{r} of intensity $\lambda_{\mathrm{r}}(r) = \lambda_{\mathrm{r}} F(r)$. Assume that Φ_{t} and Φ_{r} are independent. That is, the set of all nodes follows a PPP, e.g., a sensor network created by airdrop, and connectivity at distance d is no longer guaranteed for every node. We further assume that the routing protocol employs a nearest neighbor strategy, i.e., transmitters aim at minimizing d. For points distributed as a PPP, the CDF $F_{\mathsf{d}}(d)$ of the distance d between a point and its nearest neighbor is well-known, see [6]. We would like to know the optimal SINR threshold β such that the product $\log_2(1+\beta)\,\Omega(\beta)$ with success function $\mathbb{1}_{\{\mathsf{x} \text{ successful}\}}\mathbb{1}_{\{\mathsf{x} \text{ connected}\}}$ is maximized. This corresponds to maximizing the expected *sum rate*, i.e.,

$$\beta^* = \arg\max_\beta \left\{\log_2(1+\beta)\,\Omega(\beta)\right\}$$

$$\stackrel{(a)}{\approx} \arg\max_\beta \left\{\log_2(1+\beta)\int_0^\infty\!\!\int_0^\infty r(1 - q(r,\beta,d))F_{\mathsf{d}}(\mathrm{d}d)\mathrm{d}r\right\},$$

where we altered the notation $\Omega \to \Omega(\beta)$ and $q(r) \to q(r,\beta,d)$ to point out the functional dependencies. (a) follows from

$$\mathbb{E}\left[q(\|\mathsf{y}_i\|)|\mathsf{x}_i = x_i\right] \approx q(\|x_i\|),$$

which essentially approximates the interference field at a receiver y_i by the interference field at the associated transmitter x_i. This approximation is reasonable for high λ_{r} and/or moderate slopes of $F(r)$. Fig. 3 shows $\log_2(1+\beta)\,\Omega(\beta)$ vs. β. As can be seen, optimizing over β yields large improvements.

VI. CONCLUDING REMARKS

We extended prior work on the modeling and analysis of wireless networks by assuming an isotropic but not necessarily stationary spatial distribution of nodes. We derived, for slotted Aloha, the interference and outage statistics as a function of the receiver position and the shape of the spatial node
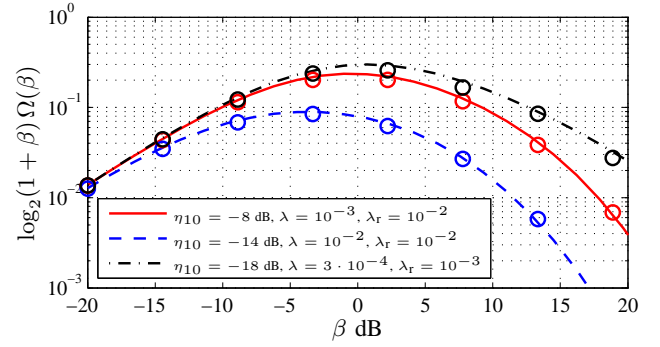

Fig. 3. $\log_2(1+\beta)\,\Omega(\beta)$ vs. β for $\alpha = 2$. η_{10} denotes η at a distance $d = 10$. $F(r) = \exp\{-r/250\}$. Marks represent simulation results.

distribution. The case $\alpha = 2$, which could not be studied yet due to the stationarity assumption, was intensively studied. For $\alpha = 4$, we also obtain closed-form results, from which known results arise as special cases. We proposed two metrics for measuring local throughput in non-stationary and finite networks and discussed possible applications of our model.

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